Solutions

61. *Find all real solutions of the following system of equations:*

$$\begin{aligned}\sqrt{x^2+y^2+6x+9}+\sqrt{x^2+y^2-8y+16} &= 5,\\ 9y^2-4x^2 &= 60.\end{aligned}$$

(50th Catalonian Mathematical Olympiad)

Solution 1 by Eloi Torrent Juste, AULA Escola Europea, Barcelona, Spain. First we observe that points (x, y) that satisfy the first equation are those that the sum of their distances to $A(-3,0)$ and $B(0,4)$ is equal to 5. Moreover, if a point P lies out of the segment AB then $AP + PB > AB = 5$. This let us to conclude that points (x, y) solution of the system must lie on AB. The equation of AB is $y = \frac{4}{3}x + 4$ or $\left(x, \frac{4}{3}x + 4\right)$ with $-3 \le x \le 0$. Substituting these values in the second equation, yields

$$9\left(\frac{4}{3}x+4\right)^2 - 4x^2 = 60 \Leftrightarrow x^2+8x+7=0$$

with roots $x = -7$ and $x = -1$. Since only the second lie in $[-3, 0]$, then the unique solution of the given system is $(-1, 8/3)$.

Solution 2 by Arkady Alt, San Jose, California, USA. Squaring both sides of the equation $\sqrt{x^2+y^2+6x+9}+\sqrt{x^2+y^2-8y+16} = 5$ we have

$$\left(\sqrt{x^2+y^2+6x+9}+\sqrt{x^2+y^2-8y+16}\right)^2 = 25 \Leftrightarrow 4x-3y+12=0$$

Then, from

$$\left.\begin{array}{c}4x-3y+12=0\\ 9y^2-4x^2=60\end{array}\right\} \Leftrightarrow \left.\begin{array}{c}3y=4x+12\\ (4x+12)^2-4x^2=60\end{array}\right\} \Leftrightarrow \left.\begin{array}{c}3y=4x+12\\ 12\,(x+7)\,(x+1)=0\end{array}\right\}$$

we obtain

$$\begin{aligned}(x,y) &= \left(-1, \frac{8}{3}\right)\\ (x,y) &= \left(-7, \frac{16}{3}\right)\end{aligned}$$

By substitution immediately follows that only $(x, y) = \left(-1, \frac{8}{3}\right)$ satisfies the given system and it is the desired solution. □

Also solved by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.

62. *Let P be an interior point to an equilateral triangle ABC. Draw perpendiculars PX, PY and PZ to the sides BC, CA and AB, respectively. Compute the value of*

$$\frac{BX+CY+AZ}{PX+PY+PZ}$$

(First BARCELONATECH MATHCONTEST 2014)